Problem Solving in Kindergarten: The Development of Children s Representations of Numerical Situations

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This paper describes the development of nine kindergarten children's representations of numerical problems over five months. The children solved a range of problems by modelling them with concrete materials, then drawing their own representations of the problem situations. The representations became more structured, with quantities delineated in various ways, including the use of letters and labels. By the end of the year, all the the children were writing their own problems and many of them were also representing problems in symbolic form.

Introduction

Representation must be considered to be a key theoretical construct in mathematics education because much of mathematics may be regarded as the representation of one structure by another and an understanding of the extent to which the original structure is preserved in the representation (Kaput, Luke, Poholsky and Sayer, 1987). The term "representation", however, is difficult to define as it can be considered to encompass physical situations, symbolic systems and mental constructs (Goldin, 1992). Until recently the importance of representations has not been emphasised in mathematics teaching despite its central role in mathematics. Lopez-Real and Vello (1993) found that the Year 5 and 6 children in their study drew diagrams for only 5% of 693 "diagram-suitable" problems. When the children were asked to draw diagrams and use these to solve problems, children gave correct solutions to approximately one third of the problems that they had previously answered incorrectly.

The relation between internal and external representations in the traditional view of mind as opposed to a constructivist approach has been explored by Cobb, Yackel and Wood (1992) who believe that mathematical meanings given to representations result from interpretation and construction by students. The notion that students develop individual interpretations of concepts, has changed the emphasis in mathematics teaching from students' answers to their solution methods. While individual perceptions may differ and representational systems may be personal, Thomas and Mulligan (1995) suggest that the further the representational system has developed structurally, the more coherent and well organised will be students' external representations.

At present little is known about the ways in which student's internal conceptions and representations generate external representations, such as pictures, during problem solving (Mariotti & Pesci, 1992). A crucial question would seem to be "What factors influence the development of student's representations and what is the teacher's role in assisting this process? The use of concrete materials has been often suggested as a way young children can model solution processes. The value of concrete representations is that they can mirror conceptual structures so a child can use the structure of the representation to construct a mental model of the concept. However, students may not see the correspondence between the structure of the material and the structure of the concept. This inability to recognise structural similarities has been suggested as a reason why concrete representations do not always assist students to learn about particular concepts (Hart, 1987; Janvier, 1987; Lesh, Landau, & Hamilton, 1983).

Perceiving and representing structural similarities requires an abstraction of the essential features of the problem and such abstractions may be difficult for students. Beveridge and Parkins (1987) found that provision of a diagram indicating a solution method was effective in helping students solve a problem when the representation was such that students recognised the structural correspondence between diagram and problem. Dubinsky (1989) suggests a major concern with visual representations, especially those that are created by students, is that "It may be true that a picture is worth a thousand words, but what if it is the wrong picture?" Key information may have been omitted or be shown in a way that students find difficult. Thus, if a representation is provided students may not recognise structural similarities between a situation and its representation, however, for students to create their own representation requires knowledge of the conceptual structure and articulation of its essential features.

One important factor in recognising and representing structural features would seem to be experience of translations among the different types of representations, as well as translations within each. This factor has been emphasised by Lesh, Landau, and Hamilton (1983). These authors consider that the act of representation may facilitate the emergence of concepts and representations during problem-solving sessions as students use different representation systems to solve problems. These authors found that in realistic problem-solving situations good problem-solvers were usually able to switch to the most convenient representation at any point in the solution process.

It is not easy to teach students to create and use representations. Two factors have been suggested as important: incorporation of numerical information from the problem into the representation and clear depiction of the relationships among problem quantities (Lopez-Real and Vello, 1993). The nature of the problem itself may also influence the representation. If students cannot draw a problem representation they may not understand its conceptual structure. Lester (1996) believes that the following results clearly emerge from the literature on problem solving:

- "Students must solve many problems to improve their problem solving ability.
- Problem solving ability develops slowly over a prolonged period of time.
- In order for students to benefit from instruction, they must believe that their teacher thinks problem solving is important.
- Most students benefit greatly from systematically planned problem solving instruction.
- Teaching children about problem solving strategies and heuristics and phases of problem solving does little to improve students' ability to solve mathematics problems in general." (p. 666)

The first two points would suggest that problem solving should start as early as possible in a child's schooling. Indeed Carpenter, T., Ansell, E., Franke, M., Fennema, E., & Weisbeck, L. (1993) showed that kindergarten children,when encouraged to model or represent actions or relationships can solve multiplication and division problems. These authors felt that consolidating and extending children's intuitive modelling skills might provide a framework for developing problem solving in the primary school. Another crucial factor may be the role of the teacher in assisting students to develop representations of mathematical structures. Lester (1996) suggests that teachers' attitudes to problem solving as well as their knowledge of planning problem solving instruction are key findings from research.

The present study builds on the findings of Carpenter et al. (1993) as it provides information about the development of kindergarten student's representations of problems over a period of approximately five months.

Methodology

The children in this study all came from one kindergarten in a medium socio-economic area of Sydney. They were separated into four ability groups at the beginning of Term 2 in Kindergarten, although these groupings were not fixed and some children later changed groups. In Terms 2 and 3 each group worked on problem solving once a week during the mathematics activity time. The groups rotated through different activities over the week; the teacher worked with the problem solving group while volunteer mothers worked with the other three groups.

The collection of the data on which this paper is based was serendipitous. The teacher of the class kept all drawings the students made in problems solving sessions and just before these were given to parents at the end of the year, the first author made copies of the annotated drawings of nine children as examples for student teachers. These included four children from the most advanced group, and two from the least advanced group. The children whose drawings were copied were selected because they used dark pencils that enabled the drawings to be photocopied. Only later were the drawings examined with the aim of examining the changes in the representations over time.

The teacher who collected the data for this study was in her first year of teaching. She has a strong commitment to developing children's independence and to involving them in thinking about their own learning. She endeavours to put these principles into practice in her teaching and exemplifies the finding reported in Lester (1996) that for students to benefit from instruction in problem solving, they must believe that their teacher sees it as important.

The contexts of the problems (see Table 1) were based on either literature read in class or familiar situations such as the children's bedrooms or the school playground. The first problems the children solved sharing situations because the initial impetus for beginning problem solving began with the nursery rhyme "Baa, Baa, Black Sheep". The teacher modelled three people and three bags of wool with different coloured cubes and said "We have three bags of wool, how many would each person get? The children replied "one", so she repeated the question for six, then nine, bags. As the children had little difficulty with this concept she asked "What if there were ten bags?" One response was to give a bag back to the sheep, the other was to share out the wool.

In the problem solving sessions the teacher began by reading the problem to the children and showing how it could be modelled with cubes. Next, children modelled the situation individually. This step was crucial because the children could not read the problem so the coloured cubes acted as a memory aide. The children were then asked to draw their own picture to show how they solved the problem. Considerable emphasis was given to showing their thinking and aspects of children's pictorial representations were discussed in their group session. Groups generally attempted the same problems although some children attempted additional problems while others did not attempt them all and required more support.

As well as problems involving repetition of a number pattern and representing three dimensional situations in two dimensions, the problems (see Table 1) included the following types (Carpenter & Moser, 1984; Carpenter et al, 1993): addition (combine); subtraction (combine and separate); multiplication (equal groups); division (partitive); and fractions (one half). After one month the teacher introduced the children to the idea of writing their own problems by asking them to make their own shopping list using a specific number pattern and imagining situations (a journey, the bedroom of their dreams). Later, as well as solving given problems, they also wrote their own problems.

Results

Since the teacher gave the children differing levels of support and there are results for only some children in the class, the number of children correctly solving each problem is not given. While the results will focus on the children's drawn representations, it should be noted that by the end of the year all the children in the class modelled the problems using concrete materials and understood the concept that one cube represented one unit.

The children's drawings developed markedly over the two terms. None of the children's initial drawings were structured but the later drawings of all children showed evidence of their solution strategies and most children's representations were increasingly organised in the way groupings were depicted.

Table 1 The problems used in the problem-solving sessions

Date	First problem	Second problem
10/5	1a Two monsters were playing, two monsters were flying, three monsters were dancing. How many monsters were at the monsters' party?	1b Four monsters are at a party. There are eight little cakes with cherries on top. How many cakes will each monster get?
22/5	2a Grandma has baked 6 cookies. Tom, Lizzie and David love her cookies. How many will they get each?	2b There were 18 cookies on the plate. Tom was very hungry and ate half of them. How many cookies were left on the plate?
29/6	3a Three dinosaurs each had two mittens. How many mittens altogether?	3b Derek the Dinosaur knitted 4 pairs of socks. How many dinosaur feet were kept nice and warm?
4/6	4a In the story "The Shopping Basket", Jeffrey had to buy lots of things. How many things did he have to buy?	4b Make up your own shopping list using the same number pattern.
11/6	5a In the story "A Lion in the Night", people chase the lion out of the castle, over the fields, into the forest, past the church, into the boat, across the sea, over the mountains and into the fields. Imagine that you are going on a journey to a magic castle. Draw a map of your journey.	
18/6	6a In "Alex's Bed", Alex rearranges his bed to make more space. Think about your bedroom and the things that are in it. Draw a plan of your bedroom.	6b Imagine that you could have the bedroom of your dreams. Draw a plan of how it would look.
23/7	7a In "Amy's Place", Amy discovers some possums investigating her new treehouse. If there were 5 mother possums, and each had three babies, how many would there be altogether?	
30/7	8a In the book "Bear and Bunny Grow Tomatoes", Bear grew lots of large, juicy tomatoes and shared them with his friend Bunny. Bear gave Bunny two boxes of tomatoes. Each box had 8 tomatoes inside. How many tomatoes did Bear give Bunny?	
6/8	9a Creepy Crawly Caterpillar had 16 legs to walk with. After he came out of the stony prison, he only had 6 legs. How many pairs of legs did Creepy Crawly Caterpillar lose?	
13/8	10a There are twelve people at Sammy's swimming party. Three are adults and the rest are children. How many people at the party are children?	10b Eight children at the party are trying not to get sunburned noses, 5 are wearing suncream and the rest hats. How many are wearing hats?
20/8	11a In her jewellery box, Janet has 6 gold rings. She also has 7 silver rings with diamonds. How many rings does Janet have in her jewellery box?	11b Make up your own problem about Janet's jewellery box.
26/8	12a One afternoon in the rainforest, there were 9 flying foxes hanging from a gum tree. Some of the flying foxes flew away to look for food, leaving four flying foxes hanging from the tree. How many flying foxes flew away?	12b Write your own problem about the KS rainforest
3/9	13a There were twelve children in the playground at lunchtime. Seven children were playing hopscotch and the rest were skipping. How many children were skipping?	13b Write your own problem about the school playground.
17/9	14a Geoffrey went to the Zoo to see all the animals. He brought along ten slices of bread for the elephants. When he had finished feeding them he had two slices of bread left. How many pieces of bread did the elephants eat?	14b Write your own problem about the zoo.
5/11	15a In the story "Alexander, Who Used to be Rich Last Sunday", Alexander spent all the money that his grandparents gave him. If Alexander had twenty dollars and he bought three lollipops that cost two dollars each, how much money would he have left?	

Figure 1 shows the change in one child's representations (Jeffrey) over four months. His first drawing has no structure, but in the second problem session he drew lines to indicate the links between cookies and people. Later drawings show increasing use of structure, for example, the array showing the five mother possums, each connected to a group of three babies for Problem 7. By session 10 he had begun to label group elements ("A" for adult) and writing equations, first with assistance (10a), then independently (10b). In an independently completed assessment task (13a) Jeffrey's drawing shows clearly separated, labelled groups as well as the corresponding equation.

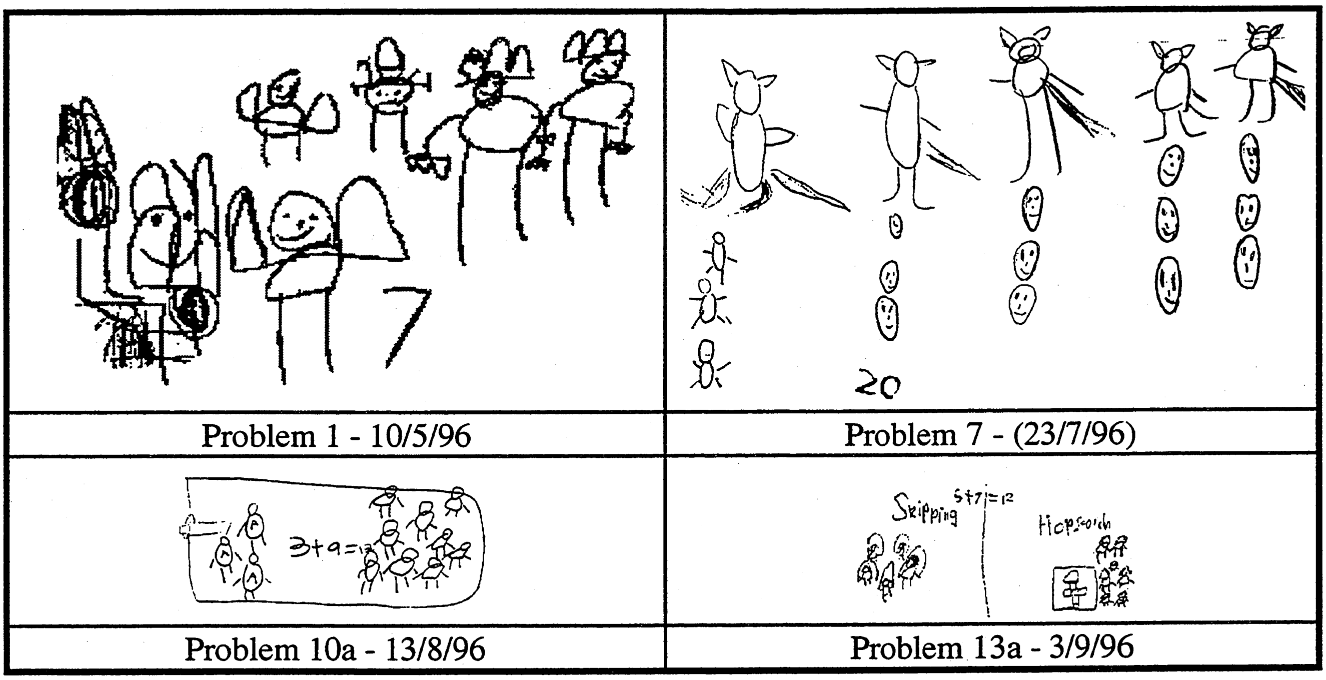

Figure 1 Jeffrey s representations of four problems

The concrete materials (cubes) that the children used to model the problems were coloured, so it would be expected that they would use colour as a means of depicting problem quantities and relationships. Indeed, use of colour was one of the first strategies children used. However, they developed a variety of other strategies in their drawings including :

- size and pictorial details (large; small; skipping ropes, etc.);
- separation for subtraction and addition;
- crossing out and partitioning of sets for subtraction;
- drawing lines to indicate sharing relationships;
- array structure to show equal groups in a multiplicative situation;
- letters and words to label elements of sets or sets.

While the use of size, pictorial details, and separation to show groupings would seem to be a natural part of drawing a picture for young children, the use of letters to label group elements or words to label the groups themselves, might not be predicted. The impetus for labelling came from one child who suggested using letters to show adults and children. This idea was adopted by a number of the other children, including Jeffrey (see Figure 1). The stimulus for this strategy may have come from the emphasis on initial sounds in early reading. It was noticeable in the drawings that children adopted representational strategies used by other children in the same group.

Use of equations

In Session 10 one child asked how equations were written using "plus" signs so the teacher worked through an example with the group and several children began writing equations, at first with assistance. Soon six of the nine children did not seem to have difficulty writing both addition and subtraction equations when these could be clearly related to the children's representations of problem structure. The children who wrote equations tended to draw groups that were delineated in some way; that is, separated, labelled, or shown with different pictorial details (e.g., figures wearing hats). Because the children were translating from a written problem to a concrete, then a pictorial and finally to a symbolic form, production of an equation was a mapping of the concrete or pictorial representation rather than of the written problem. Thus, it is not clear whether the children could have solved the equations if there was no accompanying representation. However, one child (Craig) realised that the same problem could be represented in two different ways, writing 4+5 =9 and 9-5=4 for Problem 12a. Therefore, it would seem likely that the children would be able to dispense with the intermediate representation as they gained counting skills. At this stage the children were using a count-all strategy to determine the answers to the problems.

Children had difficulty when they attempted to write equations for multistep problems such as Problem 15. The teacher did not model this problem for the group in which the children were most confident about writing equations. The children who successfully wrote equations (Craig and Briony) worked out the first step (how much the lollipops would cost) with concrete materials, marked off six dollars in their drawing, then wrote 20-6. However, children who tried to show the relation between dollars and lollipops in their drawing became confused when they tried to write the equation. They focussed on the number of lollipops and wrote the following expressions: 20+3=23; 20-3=14; and, 20-3=16.

Self-generated problems

The week after the children began writing equations the teacher asked them to write their own word problem about Janet's jewellery box (Problem 10b). For this problem all the children replicated the operation of the problem that they had just solved (Problem 10a) and wrote addition word problems. In general, the operations used in the self-generated problems were modelled on those in the immediately preceding problems. Problems 12a, 13a and 14a were not simple structures to model as the initial and final states were given and children had to determine the change. The seven children who attempted Problem 12b generated take away situations (birds flying away, koalas climbing down trees, etc.) whereas four of the five children who completed 13b wrote addition problems. Problem 14b, however, produced a variety of responses. Two children from the group that needed most support dictated addition problems:

> There was a giraffe, a tiger and a monkey. How many animals were there? (Naomi)
>
> There was a person, a parrot and a monkey walking at the zoo. How many were there? (Adam, with prompting)

Four other children modelled the same operation as the original problem (subtraction) but the structure of their problem was simpler; instead of a change situation (10-x = 2) they made the result the unknown.

> There was one lion and 10 fairy penguins. The lion ate two of the penguins. How many were left? (Craig)
>
> There were 5 kookaburras and one snake. The snake ate two kookaburras. How many were left? (Anthony)
>
> There was one monkey and 4 bananas. The zoo keeper took away one banana. How many bananas can the monkey eat? (Isabella)
>
> There were 4 birds and one lion. The lion ate one bird. How many birds were left? (Kay)

All these children included three parameters in their problems. This did not occur in earlier separate situations except for Craig who dictated the following problem about a rainforest (Problem 12b): "There was one Tasmanian tiger and five echidnas. The Tasmanian tiger ate two echidnas. How many echidnas were left?" Initially Craig wrote the equation as 6-2=4 but self-corrected this to 5-2=3. For Problem 14a he wrote 10-2= , then crossed this out and wrote 10-8=2. For his own problem about the zoo (see above) Craig confidently wrote 10-2=8. Albert and Kay also had some difficulty writing equations for their problems; Anthony first wrote 2-3=3, then crossed this out and wrote

5-2=3, while Briony wrote 4+3=3, then 4-1=3. Three children wrote their own problems and these were more complex:

> There wor 5 lin and there wor 8 tigrs tace awae too of them ho mach wor there left (David)
>
> There was 10 stars and 15 children ho memey mor stars do we need? (Alison)
>
> Threr worer five benarnas and 2 muncies. bothe of the 2 muncies aete 1 eche. How meny benarnas were left? (Jeffrey)

None the above structures had been discussed with the children: David and Jeffrey developed multistep problems: addition (David); and multiplication (Jeffrey) followed in both cases by subtraction; Alison wrote a compare subtraction structure. These children did not draw representations of their written problems, nevertheless, Alison and David attempted to write equations. David wrote 5-8=12 while Alison first wrote 10+15, then crossed it out and wrote 10-15= but she did not know how to solve her equation.

Only one child generated a combine subtraction problem in response to Problem 12a. Jeffrey wrote, "There were 2 children in the school. 6 of the 12 children were bad children and they were leifin there rubish on the playgrowid. How mene children were good?" and confidently wrote the equation as 6+6=12. The other children who attempted this problem wrote addition problems, e.g., "There was 3 peepl plaine with the hoops and 4 peepl plaine with the scipping ropes hoomen peepl wr there? 4+3=7" (Alison).

Conclusions

Although the sample is not representative, the results for these children support those of Carpenter et al. (1993) who showed that after eight months in kindergarten children who were taught problem solving could solve a variety of quite difficult word problems. The children in this study had less time spent on problem solving than the children in Carpenter et al.'s study, yet they were remarkably successful in representing and solving complex word problems, both using concrete materials and in drawings.

The children's drawings of the problem situations show that they used a variety of strategies to represent aspects of the contexts including showing properties of the problem elements (colour, size, pictorial details); separating groups or crossing out individual elements; partitioning sets and drawing lines to indicate sharing relationships; drawing array structures to show equal groups in a multiplicative situation; and using letters and words to label elements of sets or sets.

All nine children had written equations for single step problems with a direct relation between the quantities by the end of the year, and some had a good grasp of representing problems symbolically. The teacher also reported that the children became very motivated to write their own problems; most children in the class were enthusiastic about writing and painting their own problems in free time sessions.

These results suggest that both mathematics educators and teachers should question their assumptions about kindergarten children's understandings of mathematics and their skills in representing their solution processes. The results presented in this paper suggest that children can achieve a great deal when the following conditions are met:

- problem solving is seen as an enjoyable activity that is valued by the teacher;
- the teacher has an expectation that all children will benefit from problem-solving, not just those children who are seen as "bright";
- problems are linked to a familiar context;
- children model situations with concrete materials and they are encouraged to show their thinking;
- scaffolding is provided to support children at different levels; and
- problem solving skills are consolidated and extended over a period of time.

If these conditions are part of the teaching environment, then some children achieve far more than would be expected at kindergarten, both in terms of representing problem situations symbolically, and in generating problems that they have not previously encountered. The concern for young children who have been encouraged to enjoy problem solving is to ensure that their initial enthusiasm for problem solving is fostered.

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